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Teaching Transformations of Trigonometric Functions with Technology

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Trigonometry is integral to mathematics education. The field of trigonometry plays a crucial role in the study of mathematics and its applications. Despite the importance of the subject, students struggle to understand trigonometric constructs such as angle measure. It has also been noted how students struggle to understand transformations of functions generally. Our review of the literature found few studies specifically on students' understanding of transformations of trigonometric functions, but evidence exists showing students have difficulties with the concept. Here, a MATLAB program called *TrigReps* is discussed. *TrigReps* accepts four inputs for the algebraic representation *(a)sin(bx + c) + d,* and provides three additional representations as outputs. Students are presented with a graphical representation, an auditory representation, and a dynamic representation of a radius rotating around a unit circle. *TrigReps* has potential to be a useful tool for teaching transformations of trigonometric functions. In particular, it may be able to help students justify why combinations of horizontal transformations are counterintuitive. *TrigReps* is analytically sound in its design: it is interactive, dynamic, and displays Multiple External Representations (MERs) simultaneously. Initial data support its usefulness in a trigonometry classroom, but more research must be conducted to draw firm conclusions.

Introduction

Trigonometry is integral to both pure and applied mathematics education (NCTM 2000; NGA 2010). For example, trigonometric functions are necessary to derive Euler's formula, the impactful identity that states $e^{ix} = cos(x) + isin(x)$ (Stein & Shakarachi 2003). Trigonometric functions are used frequently in the STEM fields. They are present in engineering tasks such as digital image processing and finding orthogonal force vectors (Lay 2002; Rosen, Usselman, & Llewellyn 2005). Periodic behavior, such as changes in temperature or tides, also necessitates the use of trigonometric functions.

Here, a tool is presented for teaching transformations of trigonometric functions. The goal of this tool is to help students understand properties of function transformations rather than memorize them. *TrigReps*, as seen in **Figure 1**, is a dynamic, interactive computer program. *TrigReps* accepts inputs for *a*, *b*, *c*, and *d* in the function $f(x) = (a) \sin (bx + c) + d$. In response, it simultaneously provides three additional representations: a static representation of the graph on the Cartesian plane; an auditory representation of the function as a pressure wave over time; and a dynamic representation of a radius rotating around a circle. This circle has radius *a* and is centered at

(0, *d*). The radius has a starting position *c* radians from the positive horizontal axis, and it rotates counterclockwise from that position at $\frac{b}{2\pi}$ revolutions per second. While a dynamic unit circle can be used to define the sine function, no examples could be found in the literature of using one to model transformations of the sine function.

TrigReps is designed to help students justify why horizontal transformations – including combinations of horizontal transformations – are counterintuitive. Students expect addition to correspond to rightward movement, and they expect multiplication by numbers larger than one to correspond to stretching. However, this is not true. Since horizontal transformations affect the input of trigonometric functions, then, by the unit circle definitions, they affect the angle of the radius of the circle. Adding to the input rotates the starting position of the radius counterclockwise by that number of radians. This has the graphical effect of moving the original starting point behind the new starting point; the graph has moved leftward. Multiplying proportionately affects the rate at which the angle of the radius changes – i.e. the rate at which the radius rotates. Faster rotation results in a shorter period, meaning that the graph has been horizontally shrunk. Providing students with a dynamic unit circle representation while they examine transformations of trigonometric functions offers them opportunities to notice and use this information as they justify function transformation properties.
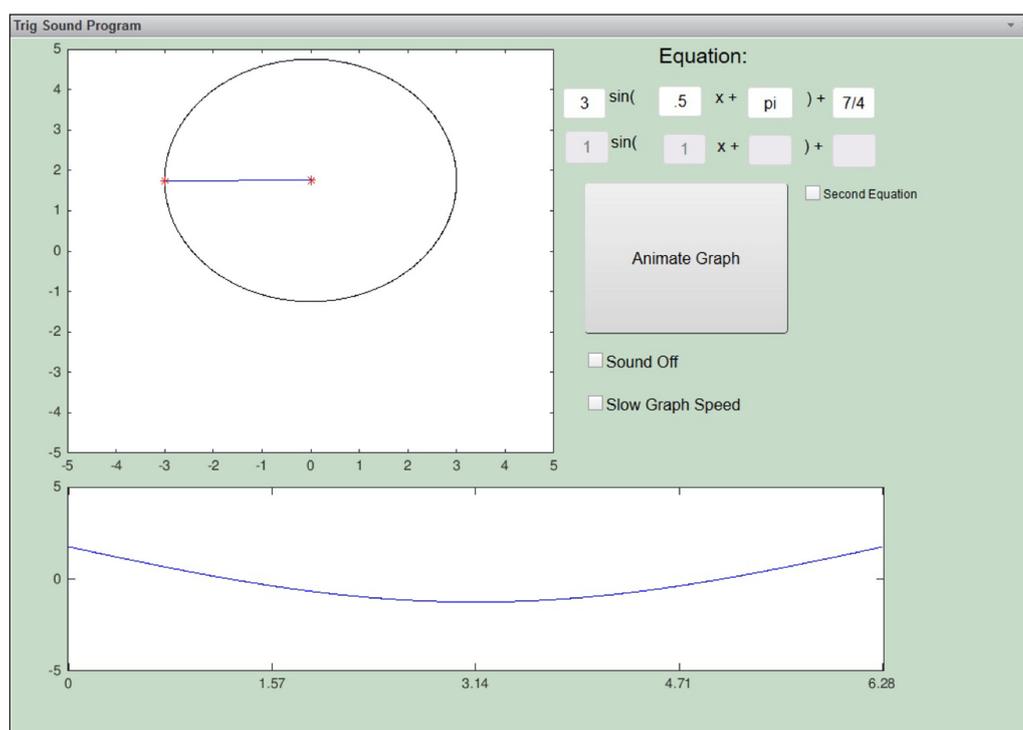
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Figure 1: A transformed sinusoid represented in TrigReps.

While additive and multiplicative horizontal transformations are individually counterintuitive, their combination is also counterintuitive. The order of operations states that multiplication is performed before addition, but additive horizontal transformations must be performed before multiplicative ones. *TrigReps* offers students the opportunity to notice that the additive transformation must be performed first since it affects the starting position of the radius. Performing the multiplicative transformation first would set the radius to rotate *b* times faster than normal. Because of this, attempting to rotate the radius by *c* radians would actually rotate it *bc* radians. Therefore, the additive transformation should be applied first. It will be noted in the literature review how issues involving horizontal transformations are or are not addressed by some of the common methods of teaching function transformations.

In addition to the literature review, a case study is presented examining how *TrigReps* helps students learn transformations of trigonometric functions. This study took place in an undergraduate precalculus classroom at a large northeastern university. The study examined written and audio data of three students using *TrigReps* to solve trigonometric problems together. The results indicate that *TrigReps* can be a useful tool for students learning trigonometric transformations. However, much more research is required to expand upon these initial studies.

In order to determine how *TrigReps* affects students' learning of trigonometric transformations, the following research questions are posed:

1) How do students interact with *TrigReps* relative to the instructors' anticipations?
2) How (if at all) does *TrigReps* illuminate students' conceptions and misconceptions about transformations of trigonometric functions?

Literature Review

In this section, a review of the literature is presented indicating that students struggle with both trigonometry and graphical transformations. Previous studies have shown that students who are unable to change between multiple representations have difficulty in trigonometry (Kendal & Stacey 1998; Weber 2005). Positive results have been seen from using dynamic, interactive technology in trigonometry classrooms (Wilhelm & Confrey 2005), especially technology that incorporates MERs (Özdemir & Ayvaz Reis 2013). Together, these results suggest that a dynamic, interactive technology which uses MERs may be helpful for students learning trigonometry.

Learning trigonometry

The existing literature on learning trigonometry raises several concerns. Numerous studies have found that students leave trigonometry classrooms with poor understanding of the subject. Studies have depicted struggles among secondary students (Kendal & Stacey 1998), undergraduate students (Moore 2013; Weber 2005) pre-service teachers (Akkoç 2008; Tuna 2013), and even in-service teachers (Topçu, Kertil, Yilmaz & Öndar 2006). Kendal and Stacey examined secondary students who had been taught trigonometry through right triangle representations and found that they had difficulty solving problems that involved non-acute angles. Weber's undergraduate participants were initially unsuccessful; they were unable to transfer their trigonometric knowledge from secondary school to the undergraduate classroom. Studies led by Moore, Akkoç, Tuna, and by Topçu each found that their participants – who had previously passed a course with a trigonometry unit – had difficulty describing radians and, more generally, angle measure. Moore's study involved undergraduate participants, but the other studies showed

pre-service and in-service mathematics teachers lacking a robust understanding of fundamental trigonometric topics.

Learning transformations

Given a function $f(x)$ and its transformation $(a) f(bx + c) + d$, students are typically asked to learn the following properties: *Outer transformations (a and d) affect the graph vertically*; *Inner transformations (b and c) affect the graph horizontally*; *Additive transformations (c and d) shift the graph*; *Multiplicative transformations (a and b) stretch/shrink the graph*; *Additive horizontal transformations behave counterintuitively*; *Multiplicative horizontal transformations behave counterintuitively*; and *Combinations of horizontal transformations are applied in a counterintuitive order*. The literature regarding students' understanding of transformations of trigonometric functions is sparse. A recent study presented a case study of a student learning horizontal transformations of trigonometric functions (Nejad 2016). This study did not examine vertical transformations or the student's conceptions of the similarities and differences among various transformations. Rather, it focused on using the period of the transformed function to find the algebraic representation. The student struggled with transformations that involved fractional coefficients. These coefficients caused her difficulty finding the transformed periods, which led to incorrect conceptions of the effect of the transformations.

Apart from trigonometry, numerous researchers have stated that students tend to experience difficulty learning graphical transformations generally (Barton 2003; Borba & Confrey 1996; Faulkenberry & Faulkenberry 2010; Hall & Giacin 2013). Faulkenberry and Faulkenberry describe a phenomenon that is noted in each study, "Transformations to the input tend to give students more difficulty conceptually, most likely because the effects of transformations on the input seem to be counterintuitive" (p. 30). Developing a robust understanding of horizontal transformations has proven so difficult that it is not unusual for students to be encouraged to notice the patterns of transformations and simply memorize that the behavior of horizontal transformations is counterintuitive (e.g. Axler 2013; Barton 2003)

Strategies have been suggested to help students justify the nuances of graphical transformations instead of memorize them (Borba & Confrey 1996; Hall & Giacin 2013). The methods presented by previous researchers each have strengths and weaknesses. The *rubber sheet method* characterizes horizontal transformations as affecting the x- and y-axes rather than the function curve, as seen in **Figure 2** (Borba & Confrey 1996; Faulkenberry & Faulkenberry 2010). This method makes single horizontal transformations intuitive: moving the axes to the right has the same effect as moving the curve to the left, while stretching the axes has the same effect as shrinking the curve. However, combinations of horizontal transformations remain counterintuitive. Using the rubber sheet method, additive transformations must still be applied before multiplicative ones.

Hall and Giacin (2013) have proposed the *horseshoe method*, as seen in **Figure 3**, which gives an explanation for all counterintuitive aspects of horizontal
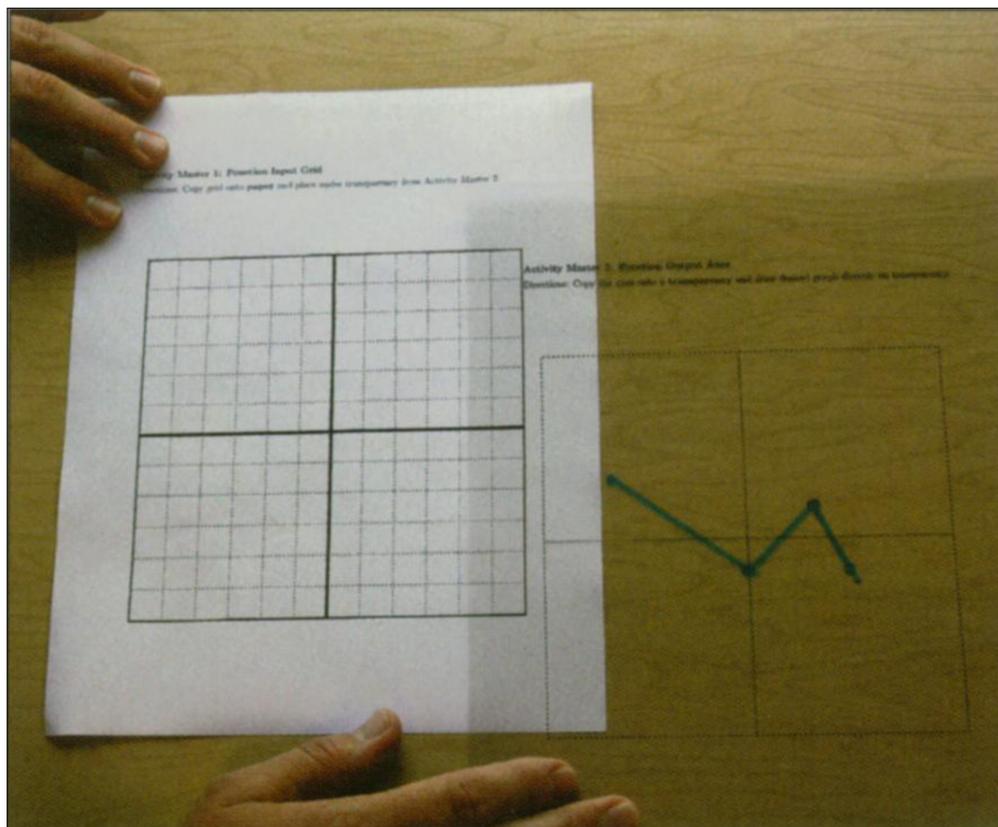

Figure 2: The coordinate axes and the curve as separate objects (Faulkenberry & Faulkenberry, 2010).

$$\left(x, f\left(x\right)\right) \xrightarrow{T} \left(x', f\left(x' - 3\right)\right)$$
$$\uparrow \qquad\qquad \downarrow$$
$$y = f\left(x\right) = x^2 \qquad g\left(x'\right) = \left(x' - 3\right)^2$$

Figure 3: The horseshoe method shifting the function $y = x^2$ three units to the left (Hall & Giacin, 2013).

transformations. They advise manipulating a horizontal transformation such as $f(x + 3)$ via the substitution $x' = x + 3$ so that the ordered pair $(x + 3, f(x))$ becomes the ordered pair $(x', f(x'-3))$. Using the horseshoe method, shifting to the left is represented with subtraction, which is more intuitive for learners. Combinations of transformations are also made to appear more intuitive after applying the horseshoe method. For example, the ordered pair $(2x + 3, f(x))$ becomes the ordered pair $\left(x', f\left(\frac{x'-3}{2}\right)\right)$. In this notation, the order of operations instructs that the additive transformation is applied to x' before the multiplicative one. While this method does present horizontal transformations in a notation that makes their application and combination intuitive, the notation is cumbersome and students must still memorize that they need to apply this method.

Learning with MERs
Students learning horizontal transformations may be aided by utilizing MERs. Weber (2005) argues that the ability to fluidly change representations is integral to learning trigonometry. Some problems cannot be approached productively using a single representation. For example, students may have difficulty justifying that $cos(-\theta) = cos(\theta)$ without moving from the algebraic representation to the graphical or unit circle representations. The research conducted by Kendal and Stacey (1998) and by Delice and Roper (2006) each examine students taught primarily using a single type of external representation. Each examined cohort had difficulty solving problems that were presented using an unfamiliar representation.

In contrast, dynamic and interactive representations have been effective in trigonometry classrooms (Kessler 2007; Rosen et al. 2005; Sokolowski & Rackley 2011; Wilhelm & Confrey 2005; Zengin, Furkan, & Kutluca 2012). Studies performed by Kessler, by Rosen and colleagues, and by Wilhelm and Confrey have examined trigonometry classrooms that utilized dynamic, interactive representations while studying sound waves. These lessons were reported to be generally effective for improving students' understanding of the trigonometric functions. Some researchers theorize that students are willing to participate more fully in class activities centered around sound wave applications than pure mathematical activities because of the students' capabilities to connect the material to their lives outside of the classroom (Douglas, Christensen, & Orsak 2008; Kessler 2007; Rosen et al. 2005).

One limitation of using technology in the classroom is that students may attempt to use it to replace actively thinking about the mathematics (Ellington 2003). Rosen and colleagues (2005) and Nejad (2016) performed studies in which students were allowed to use a computer program to help them solve trigonometry problems. The students were generally able to perform the tasks proficiently and arrive at the correct answer, but when asked later to reflect upon why their responses were true, the students were unable to justify their reasoning other than to say that the computer confirmed that their answers were correct. In contrast, several studies have examined students making useful inferences in similar situations (Sokolowski & Rackley 2011; Wilhelm and Confrey 2005; Zengin et al. 2012). In these studies, students were guided through tasks that asked them to explore representations, make and test hypotheses, and generalize their thoughts about the sinusoidal functions. Through this reflection, students discovered several properties of trigonometric functions. In each study, the authors indicate that the students were generally able to provide accurate, justified responses during their assessments, which indicates that reflective tasks must be paired with technology to ensure that students do not use it mindlessly.

A pilot study for *TrigReps* demonstrated that the program could be effective at helping students notice connections among representations (Bornstein 2017). Using *TrigReps*, students were asked to find functions with certain characteristics such as an amplitude of 0.2, or a frequency twice that of $sin(x)$. Following the activity, students' work was examined. All twelve participants described how changes in the algebraic representations corresponded with changes in the graphical representations. Another eight students also included the audio representation, and three of those students wrote of connections among all of the representations. While the tasks were effective at getting students to notice connections between at least two representations, the researchers recommended revising the tasks so that students would be prompted to notice connections among all of the representations.

TrigReps is a dynamic, interactive program that displays MERs. When paired with a set of tasks that ask students to reflect upon their mathematical activity, *TrigReps* has the potential to help students learn transformations of trigonometric functions. In particular, the dynamic unit circle representation displayed by *TrigReps* provides students with a structure to facilitate the development of their understanding of horizontal transformations.

Conceptual Framework
To examine the efficacy of *TrigReps*, Mason's (2008) *shifts of attention* framework will be utilized. This framework has five stages, which describe how students progress from simply being aware of a mathematical construct to actively using the construct to examine mathematical relationships. Mason describes the state of being aware of a construct as *holding wholes*. During this stage, the student has not begun to think about properties, implications, or anything specific. Mason notes that this stage may last for only a minuscule period of time before the student begins to notice specifics or reflect on their previous experiences. Once the student becomes aware of particulars, they have

entered the *discerning details* stage. In the context of the unit circle, the student is *holding wholes* when they are aware of the circle; they are *discerning details* when they notice, for example, that the radius is one, or that the circle is centered at the origin. The third stage, *recognizing relationships*, occurs when the student becomes aware of properties or relationships based upon previous experience. These relationships are what Tall and Vinner (1981) refer to as a concept image. For example, the unit circle may cause a student to think of trigonometric functions, angle measure, the value 2π, and relationships among these concepts. These first three stages can occur very quickly. Mason even describes the process of recognizing relationships as occurring "automatically" (p. 37).

The final two shifts of attention are more active and time-consuming. Once students have enough details, they can begin noticing patterns and testing hypothesized properties. This is what Mason (2008) refers to as *perceiving properties*. For example, while examining the unit circle, students may notice that a diameter of the circle intersects at opposite x- and y-values. That is, if one intersection is the ordered pair $\left(\frac{3}{5}, -\frac{4}{5}\right)$, the other intersection will be $\left(-\frac{3}{5}, \frac{4}{5}\right)$. In terms of trigonometric functions, this can be expressed as the identities $cos(x + \pi) = -cos(x)$ and $sin(x + \pi) = -sin(x)$. Finally, students using their hypotheses to predict additional mathematics are said to be *reasoning on the basis of perceived properties*. This stage requires students to extend their hypothesized properties to new situations. Students may develop new ideas from their properties, such as reasoning that $cos(x + 2\pi) = cos((x + \pi) + \pi) = -cos(x + \pi) = cos(x)$ or, simply applying the property to a new example also demonstrates that the student has reasoned based on their properties.

Once students have been introduced to the mathematical tasks, they have begun holding wholes. Using *TrigReps*, students have the opportunity to discern details from several representations simultaneously. *TrigReps* performs all of the calculations to plot a transformed function, play the tone that corresponds to the frequency of the sine wave, and animate a circle, which each illustrate the effects of the transformations. Using these details, students can begin the process of perceiving properties by hypothesizing and testing patterns. To ensure that students do not use *TrigReps* to mindlessly guess and check, tasks that prompt students to actively hypothesize should be used (cf. Nejad 2016; Rosen et al. 2005). By asking students to actively make connections among representations, they should have more opportunities to reason based on their understanding.

Methods

Bornstein's (2017) pilot study demonstrated that *TrigReps* can be used to successfully direct students' attention towards the given representations of transformations of trigonometric functions. It must still be determined how to capitalize on that attention to help students learn. To that end, a qualitative case study was designed to investigate more specifically how students use *TrigReps*. This methodology is appropriate for an open-ended examination of how students use the program.

Data collection

This qualitative case study examines the work of a group of three students in an undergraduate precalculus class at a large, northeastern university. The group consisted of three women, Alexa, Brianna, and Caitlin (the names given here are pseudonyms). Each of the three were freshmen. Alexa and Brianna were majoring in business administration, while Caitlin was studying marine biology. The class was presented with a fifty-minute lecture covering transformations of trigonometric functions. The following class, students were assigned a worksheet to complete in groups of three or four. Each group was provided with a laptop computer already powered on and running *TrigReps*. While they worked, the participants' conversation was audio recorded. Afterwards, their written work was collected and photocopied.

Data analysis

The students' group work was transcribed and coded. The initial codes noted which representations students were given, which representations they used, whether they used ratio or unit circle definitions of trigonometric functions, if they were incorrect, and the particular strategies they employed. Common strategies included using reference angles on the unit circle, memorized outputs of trigonometric functions, the Pythagorean identity $cos^2(\theta) + sin^2(\theta) = 1$, and the special 30-60-90 and 45-45-90 right triangles.

The data was additionally coded according to Mason's (2008) *shifts of attention* framework. It was assumed that students would be *holding wholes* at the beginning of the activity. This stage describes students who are aware of being in a mathematical situation but who have not begun to examine any particulars. When the students began to notice particulars, their statements were coded as *discerning details*. Examples include describing the amplitude of a particular function, noting an ordered pair on a graph, or characterizing the pitch of the audio representation.

Students often used details that they did not discern from the representations given to them. When students utilized their prior knowledge, it was coded as *recognizing relationships*. Examples include using identities, facts about the ranges of trigonometric functions, or right triangle trigonometry. This code was meant to capture all of the trigonometric conceptions that students had noticed before starting the activity.

The code *perceiving properties* was used to identify when students described trigonometric concepts that were new to them, namely relationships among transformations of the algebraic, graphical, unit circle, and audio representations. It was hoped that the learning goals for the activity would be among the properties perceived by the students, but these were not the only statements that could be coded this way. The activity was intended to help students learn connections between algebraic and graphical representations of trigonometric transformations, but students also had the opportunity to notice, for example, that amplitude correlates with volume. In addition to coding when students stated an awareness of new properties, it was also noted when students applied *reasoning on the*

basis of perceived properties. This stage indicates that the student has learned the concept well enough to apply it to a new situation. It was intended that students would be able to reason on the basis of connections they found between the algebraic and graphical representations of trigonometric functions to predict how changes to one representation would affect the other.

Results

Alexa, Brianna, and Caitlin were able to successfully complete the set of tasks that examined single transformations. However, there was not enough time for them to begin the set of tasks that examined combinations of transformations. Discussion amongst the group showed that the students made predictions regarding function transformations, used *TrigReps* to examine the transformations, and reflected on their predictions. For written work, the group submitted paraphrases of their discussion.

Part of the first task involved finding a function with twice the amplitude of $f(x) = sin(x)$. The following remarks were made:

> **Alexa:** For twice the amplitude, do we just do $2sin(x)$?
> **Caitlin:** That made the amplitude greater. The peaks are taller.
> **Alexa:** Why isn't it dinging though?
> **Brianna:** Because the frequency is still the same. It needs to have a lot more of the loop-dee-loops in the same 2π. I'm just going to write $f(x) = 2sin(x)$. $2sin(x)$ made it just taller. And the 0.2 graph made it…. It's just how tall it is.
> **Alexa:** It's the amplitude.

Later, when they were examining the effects of transformations on the audio representation, they began with input $sin(400x)$ and had this exchange:

> **Brianna:** That's a cool graph.
> **Alexa:** Why does it look like that?
> **Brianna:** It made a sound at the beginning like when you plug a guitar into an amp…. What if we did amplitude as well as [input $c = 400$]? I heard that a lot clearer. I didn't touch the sound and it got so much louder. Let's try —
> **Alexa:** Aaaaaaah!
> **Brianna:** That did something. Let's not try 100. Let's go back to 10. We did an amplitude of 100 and a frequency of 400. We won't do that anymore…. I guess that's why it's called an amp.

The group noticed that when they changed the a-input of *TrigReps*, it also caused the amplitude of the graph and the volume of the audio to change.

The group spent very little time on the set of tasks examining vertical shifts. While looking for a function that shifted $f(x) = sin(x)$ down by π, Brianna said to "put negative π in the d slot," meaning the fourth input for $\_sin(\_x + \_) + \_$ in *TrigReps*. They immediately turned their attention to shifting $f(x) = sin(x)$ up by $\frac{3}{2}$ units. Brianna continued,

"up by three halves. Instead of negative π, it would be plus three halves. Woohoo!" In their written work, they also noted the effect on the unit circle. They wrote that "changing values outside the parentheses moves the graph and unit circle up and down."

Horizontal shifts were slightly more difficult. While trying to shift the graph of $sin(x)$ to the left by $\frac{\pi}{2}$, they input $sin\left(x - \frac{\pi}{2}\right)$ and made the following remarks:

> **Brianna:** Did that make it go to the right? So plus?
> **Caitlin:** It would be $sin\left(x + \frac{\pi}{2}\right)$.
> **Brianna:** …and to the right by seven. Why did that shift?
> **Caitlin:** What?
> **Brianna:** The radius went to a different spot.

Although Brianna commented on the shift of the radius, the group did not explore that phenomenon, and it was not remarked upon in their written work.

The next task asked them to find a function with triple the frequency of $sin(x)$. They input $sin(3x)$ and Caitlin noted that "for $sin(3x)$, the frequency increases." The group wrote, "We decided to do $3x$ and we saw three humps." Continuing, they hypothesized that the function $sin(440x)$ would have frequency 440 Hz. However, while the frequency is 440 times that of $sin(x)$, the frequency of $sin(x)$ is only $\frac{1}{2\pi}$, so their function actually produced a sound that was about 70 Hz. Brianna exclaimed "Oh, that was so low! It's like a little hum! It sounds like a little submarine." She also noted that "as the frequency goes up, [the graph] looks more and more like a blue, gigantic caterpillar. I'm legitimately writing that." In the written work, the group summarized, "Increasing the frequency increases the pitch, the number of cycles per second on the graph, [and] the speed of the radius on the unit circle," and "Changing values within the sine parentheses shifts the graph left or right."

This group was able to complete the tasks that addressed each of the individual transformations: vertical stretches, vertical shifts, horizontal stretches, and horizontal shifts. The group did not have enough time to finish the activity. The remainder of the tasks examined combinations of transformations. An additional class period would have been necessary for students to work through these concepts.

Discussion

Recall Mason's (2008) *shifts of attention* framework. The students were assumed to be *holding wholes* at the beginning of the class period; they knew they were taking part in a mathematical activity. Working as a group, the students *discerned details*, including the shape of the graph, the volume and pitch of the sound wave, and the placement of the unit circle. *Recognized relationships* refers to the prior knowledge that the students associated with the given information, such as volume, pitch, and the unit circle definition of sine. The goal of the activity was for students to *perceive properties* about graphical transformations of trigonometric functions, such as "addition correlates with shifts," and to *reason on the basis*

of perceived properties. After perceiving a new property, was the group able to predict the effects of applying the property?

Details discerned by the group include descriptions of representations such as when Caitlin said "That made the amplitude greater. The peaks are taller." Also included were observations like "I heard that a lot clearer," and "The radius went to a different spot." *TrigReps* aided this process by providing the representations for the group to examine. The students could have plotted the graphs given enough time, and *TrigReps* saved them time by automating the process. The students had not seen the dynamic unit circle or audio representations in class previously. *TrigReps* offered them opportunities to notice connections among each of the new representations and the representations that they were familiar with.

The group needed to recognize some relationships in order for *TrigReps* to be effective. For example, if students don't know the unit circle definition of sine, they will not be able to make connections to the unit circle representation. Going into the activity, it had been explained to the students that the graphical, dynamic unit circle, and audio representations would respond to their inputs for the algebraic representation. The class had been taught both the unit circle and ratio definitions of the sine function. Through the unit circle, they had seen that sine behaves periodically and is defined for all inputs. It can be seen that the group was aware of these facts through comments about the frequency. When they input $sin(3x)$ and wrote "We decided to do $3x$ and we saw three humps," They knew that they would see more than a single, horizontally shrunken cycle. They knew that sine would repeat itself and that they could increase the number of repetitions they would see.

There is evidence that the group perceived a number of properties for single transformations. Given $asin(bx + c) + d$, the work from the group implies that they have learned that the outer transformations (a and d) affect the graph vertically and inner transformations (b and c) affect the graph horizontally. This can be seen in their statements, "Changing values outside the parentheses moves the graph… up and down," and "Changing values within the sine parentheses shifts the graph left or right." They have learned that multiplicative transformations (a and b) stretch/shrink the graph and additive transformations (c and d) shift the graph; they found that "for $sin(3x)$, the frequency increases," and that the graph was shifted left and right by $sin\left(x + \frac{\pi}{2}\right)$ and $sin\left(x - \frac{\pi}{2}\right)$, respectively. From this last fact, they also learned that additive horizontal transformations behave counterintuitively.

While their work is consistent with having learned that multiplicative horizontal transformations behave counterintuitively, they referred to these transformations in terms of the frequency of the function, rather than the period. Since the frequency is the inverse of the period, the terminology is not counterintuitive; instead of saying that multiplying by large numbers shrinks the function, they said that it increases the frequency. Because of this, they never commented on the difference between vertical multiplicative transformations and horizontal ones.

Finally, the group did not have time to examine combinations of transformations.

The final stage of Mason's (2008) framework, reasoning on the basis of perceived properties, shows that the student is able to use their new knowledge to predict future outcomes.

Each task had multiple parts so that students could make and test hypotheses. After finding that $2sin(x)$ had twice the amplitude of $sin(x)$, the group reasoned that $0.2sin(x)$ would have 0.2 amplitude. Their inclination to change d to negative π in order to shift the graph down vertically indicated that they had reasoned that the outer transformations affected the graph vertically. Furthermore, they used multiplication to increase the frequency, indicating that they had associated that operation with stretching/shrinking the graph. After they tried to shift to the left using subtraction, they corrected themselves and accurately predicted the solutions for the remaining horizontal shift tasks.

It cannot be determined exactly which of the learning goals were newly perceived properties and which were recognized relationships that the students were familiar with to start the activity. That being given, by the end of the activity, the students demonstrated that they could predict the behavior of single transformations. *TrigReps* offered the group opportunities to notice connections among the algebraic, graphical, unit circle, and audio representations of transformations of the sine function. The group was able to utilize these representations to make productive observations about graphical transformations of trigonometric functions.

Conclusion

Out of the seven learning goals for function transformation, the group demonstrated that they had learned at least five. They showed that they understood the effects of multiplicative transformations and additive transformations. They also showed that they understood the effects of transformations inside and outside of the parentheses. Additionally, the group noted that additive horizontal transformations behave counterintuitively. There is not enough evidence to show that they understood the counterintuitive behavior of multiplicative horizontal transformations, and the group did not examine combinations of transformations.

This study demonstrated that *TrigReps* can be a useful tool to teach transformations of trigonometric functions when used in conjunction with classroom activities that promote inquiry and reflection. After using *TrigReps*, students made productive observations regarding the behavior of graphical transformations. The representations provided by *TrigReps* helped the students to make connections between their changes to the algebraic representation and the resulting changes to the graphical representation.

The audio representation provides students with another way to connect their experiences with the mathematics. Students are familiar with the concepts of sound volume and frequency, and *TrigReps* gives them opportunities to notice how those concepts are related to transformations

of the sine function. Future research could examine how the inclusion of the audio representation in *TrigReps* affects how students learn graphical transformations of trigonometric functions.

In addition to providing another representation for students to connect with, the audio representation could affect students' motivation. In this study, group members expressed excitement about the audio representation. Previous researchers have indicated that they believe students are more motivated learning trigonometry with auditory applications than with pure mathematics (Kessler 2013; Rosen et al. 2005; Wilhelm & Confrey 2005). More research is necessary to determine whether these beliefs are supported by evidence, and *TrigReps* could be used for such research.

This study was limited by its short implementation period. Future studies of trigonometric transformations using *TrigReps* could be conducted over a longer period of time. They could also examine if students are able to transfer what they learn using *TrigReps* to other contexts. That is, after using the program, can students perform trigonometric tasks independently? How do students utilize their knowledge of the auditory and dynamic unit circle representations on future tasks? While this study demonstrates that *TrigReps* has potential as a tool for teaching trigonometry, more research is necessary.

Competing Interests
The author has no competing interests to declare.

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